On a formula for the Lusztig induction over the unipotent representations of $\mathrm{O}_{2n}^{\pm}(\mathbb{F}_q)$

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Bipartitions

Partition of m: Multiset of non-increasing positive integers $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$ such that $|\lambda| := \lambda_1 + \cdots + \lambda_r = m$.
Partitions are represented by **beta-sets**:
$\beta(\lambda) := \{\lambda_1, \lambda_2 - 1, \dots, \lambda_r - (r-1), -r, -r-1, \dots\} \subset \mathbb{Z}$.

We will work with shifted beta-sets: $\beta_d(\lambda) = \beta(\lambda) + d$, $d \in \mathbb{Z}$.

Bipartitions of m: pairs of partitions (λ^1, λ^2) satisfying $|\lambda^1| + |\lambda^2| = m$. We may shift each partition, and represent them by pairs of beta-sets:

$$\begin{pmatrix} \beta_{d_1}(\lambda^1) \\ \beta_{d_2}(\lambda^2) \end{pmatrix} =: \begin{pmatrix} X \\ Y \end{pmatrix}.$$

Symbols and ordered symbols

Symbols: Equivalence classes of (shifted) bipartitions generated by the relations:

$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim \begin{pmatrix} X+d \\ Y+d \end{pmatrix} \text{ and } \begin{pmatrix} X \\ Y \end{pmatrix} \sim \begin{pmatrix} Y \\ X \end{pmatrix} =: \begin{pmatrix} X \\ Y \end{pmatrix}^{\dagger}$$

Ordered symbols: Equivalence classes only by shifts.

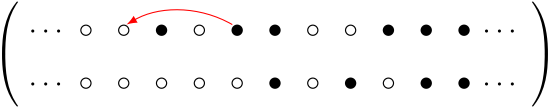
We will also represent symbols with lines of dots. Example: The bipartiton $(\beta_1(3,2,2), \beta_{-1}(2,1))$ is represented by:

$$\begin{pmatrix} \cdots & \circ & \circ & \overset{4}{\bullet} & \circ & \overset{2}{\bullet} & \overset{1}{\bullet} & \circ & \circ & \bullet & \bullet & \bullet & \cdots \\ \cdots & \circ & \circ & \circ & \circ & \circ & \bullet & \circ & \bullet & \circ & \bullet & \bullet & \cdots \end{pmatrix}$$

e-hooks and e-cohooks

***e*-hook**: an element $x \in X$ (or $y \in Y$) such that $x + e \notin X$ ($y + e \notin Y$).

Add an *e*-hook: find an e-hook x and change X by $(X \backslash \{x\}) \sqcup \{x + e\}$; analogously for Y.



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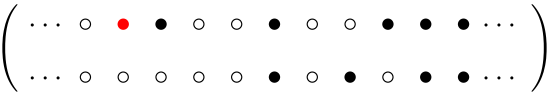
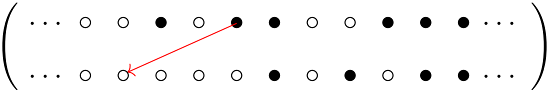


e-**cohook**: an element $x \in X$ (or $y \in Y$) such that $x + e \notin Y$ ($y + e \notin X$). **To add an** *e*-**cohook** x consists on changing X by $X \backslash \{x\}$ and Y by $Y \sqcup \{x + e\}$; or viceversa.



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Unipotent representations of $SO^{\pm}_{2n}(\mathbb{F}_q)$

Let q be a power of an odd prime and $n \geq 0$. Let $\eta \in \{+, -\}$. The unipotent representations of $SO^{\eta}_{2n}(\mathbb{F}_q)$ are parametrized by symbols of the form

$$\begin{pmatrix} \beta_t(\lambda^1) \\ \beta_{-t}(\lambda^2) \end{pmatrix},$$

with $t \geq 0$ and λ^1, λ^2 partitions such that $t^2 + |\lambda^1| + |\lambda^2| = n$.

$SO^{+}_{2t^2}(\mathbb{F}_q)$ (resp. $SO^{-}_{2t^2}(\mathbb{F}_q)$) has a cuspidal unipotent representation for every even (resp. odd) number $t \geq 0$. It is parametrized by the symbol above with $\lambda_1 = \lambda_2 = \emptyset$.

Problem: each symbol with $\lambda_1 = \lambda_2$ and $t = 0$ parametrizes two unipotent representations! We call them degenerate symbols.

Induction of Deligne-Lusztig on the special orthogonal groups

Theorem (Asai '84, Malle '20)

Let $n, e \in \mathbb{Z}$ with $e \geq 1$ and $n \geq 0$. Let $G^\circ = \mathrm{SO}^{\eta}_{2n+2e}(\mathbb{F}_q)$. Let L° be a Levi subgroup of G° isomorphic to $\mathrm{SO}^{-\eta}_{2n}(\mathbb{F}_q) \times T$ with $|T| = q^e + 1$. Let $\tilde{\Lambda}$ be a symbol as before, and $\rho_{\tilde{\Lambda}}$ the unipotent representation of L° labelled by $\tilde{\Lambda}$ (or the sum of two such representations, if $\tilde{\Lambda}$ is degenerate). Then

$$R^{G^\circ}_{L^\circ}(\rho_{\tilde{\Lambda}}) = \sum_{\tilde{\Lambda}'} \pm \rho_{\tilde{\Lambda}'},$$

where the sum runs over the symbols $\tilde{\Lambda}'$ obtained by adding an e-cohook to $\tilde{\Lambda}$. The sign changes if we take an adjacent e-cohook.

A reminder of Clifford theory

$SO^{\eta}_{2n}(\mathbb{F}_q)$ is a subgroup of index 2 of $O^{\eta}_{2n}(\mathbb{F}_q)$. Take $\tilde{\rho} \in \mathrm{Irr}(SO^{\eta}_{2n}(\mathbb{F}_q))$. Either

- $\rho := \mathrm{Ind}^{O}_{SO}(\tilde{\rho})$ is irreducible in $O^{\eta}_{2n}(\mathbb{F}_q)$, and therefore $\mathrm{Res}^{O}_{SO}(\rho)$ is the sum of $\tilde{\rho}$ and other irreducible representation of SO_{2n}, or
- $\mathrm{Ind}^{O}_{SO}(\tilde{\rho})$ is a sum of two irreducible representations ρ_1 and ρ_2, and $Res^{O}_{SO}(\rho_i) = \tilde{\rho}$.

Unipotent representations of $O^{\pm}_{2n}(\mathbb{F}_q)$

$\rho \in \mathrm{Irr}(O^{\eta}_{2n}(\mathbb{F}_q))$ is called unipotent if the restriction to $SO^{\eta}_{2n}(\mathbb{F}_q)$ is factorized by unipotent representations of SO.

Let Λ be an ordered symbol; we call $\tilde{\Lambda}$ the corresponding symbol.

- If $\tilde{\Lambda}$ is a non-degenerate symbol as before, $\mathrm{Ind}^{O}_{SO}(\rho_{\tilde{\Lambda}})$ is a sum of two representations. We will label them by Λ and $\Lambda^{\dagger}$.
- If $\tilde{\Lambda}$ is degenerate, the induction of each representation labelled by it is irreducible. We label this representation by Λ.

Therefore, we can label the unipotent representations of $O^{\eta}_{2n}(\mathbb{F}_q)$ by the following ordered symbols:

$$\begin{pmatrix} \beta_t(\lambda^1) \\ \beta_{-t}(\lambda^2) \end{pmatrix}.$$

Problem: We must make a choice for each pair of representations labelled by a symbol and its opposite.

Conjectural formula for the orthogonal groups

Conjecture

Let $n, e \in \mathbb{Z}$ with $e \geq 1$ and $n \geq 0$. Let $G = O^{\eta}_{2n+2e}(\mathbb{F}_q)$. Let L be a Levi subgroup of G isomorphic to $O^{-\eta}_{2n}(\mathbb{F}_q) \times T$ with $|T| = q^e + 1$. Then, there exists a parametrization of the unipotent representations of the even orthogonal groups such that, if ρ is a unipotent representation of L and Λ is the symbol labelling ρ, we have that

$$R^G_L(\rho) = \sum_{\Lambda' \setminus \Lambda = e\text{-cohook}} \pm \rho_{\Lambda'}.$$

The sign changes if we take an adjacent e-cohook.

Sketch of the progress

We rename $\rho_\Lambda := \rho$. We recall that

$$R_{\mathbf{L}^\circ}^{\mathbf{G}^\circ}(\rho_{\tilde{\Lambda}}) = \sum_{\tilde{\Lambda}' \setminus \tilde{\Lambda} = e\text{-cohook}} \pm \rho_{\tilde{\Lambda}'}. \tag{1}$$

Digne-Michel '94:

$$\mathrm{Res}^{G}_{G^\circ} \circ R^{\mathbf{G}}_{\mathbf{L}} = R^{\mathbf{G}^\circ}_{\mathbf{L}^\circ} \circ \mathrm{Res}^{L}_{L^\circ}.$$

Therefore, we have that:

$$R^{\mathbf{G}}_{\mathbf{L}}(\rho_\Lambda) = \sum_{\Lambda' \setminus \Lambda = e\text{-cohook}} \pm \rho_{\Lambda'^*} + \sum_{\Theta} a_\Theta(\rho_\Theta - \rho_{\Theta^\dagger}),$$

where $\Lambda'^* \in \{\Lambda', \Lambda'^\dagger\}$.

Induction on $n+e$: The formula is true for $q > 2(n+e)$ by Waldspurger. So the case $n+e=1$ is covered, since $q \geq 3$.

For the residual term, we calculate

$$\left\langle R_{\mathbf{L}}^{\mathbf{G}}(\rho_\Lambda), R_{\mathbf{L}}^{\mathbf{G}}(\rho_\Lambda)\right\rangle = \left\langle \rho_\Lambda, {}^*R_{\mathbf{L}}^{\mathbf{G}}(R_{\mathbf{L}}^{\mathbf{G}}(\rho_\Lambda))\right\rangle$$

This term disappears for $n=0$ thanks to the Mackey formula for tori, and because the trivial representation is labelled by a degenerate symbol.

From now on, we assume $n+e \geq 2$ and $n \geq 1$. We assume that the formula is true for any n', e' with $n'+e' < n+e$.

Assuming the Mackey formula for (L, L), we can calculate the scalar product thanks to the induction hypothesis, and it will result on the annulation of the extra term.

And, if Λ is degenerate, the result is true, no matter the parametrization. We can assume then that Λ is not degenerate.

We have

$$R_{\mathbf{L}}^{\mathbf{G}}(\rho_\Lambda) = \sum_{\Lambda' \backslash \Lambda = e\text{-cohook}} \pm \rho_{\Lambda'^*}$$

We fix Λ_{Ind} a symbol over Λ. To determine which one between $\rho_{\Lambda'}$ or $\rho_{\Lambda'^{\dagger}}$ occurs, we search for a Λ_{Res} that gives us Λ_{Ind} after adding a 1-hook, and no other Λ' over Λ can be obtained like this.

Our goal: We consider $M = \mathbb{F}_q^{\times} \times O^{\eta}_{2n+2e-2}(\mathbb{F}_q)$. Then $\rho_{\Lambda_{\mathrm{Res}}}$ is a representation of M, and $R^G_M(\rho_{\Lambda_{\mathrm{Res}}}) = \rho_{\Lambda_{\mathrm{Ind}}} +$ other representations not parametrized by any Λ'. So $\Lambda_{\mathrm{Ind}} | R^G_M(\rho_{\Lambda_{\mathrm{Res}}})$, and

$$\begin{aligned} \left\langle R^{\mathbf{G}}_{\mathbf{M}}(\rho_{\Lambda_{\mathrm{Res}}}), R^{\mathbf{G}}_{\mathbf{L}}(\rho_{\Lambda}) \right\rangle &= \left\langle \rho_{\Lambda_{\mathrm{Res}}}, {}^*R^{\mathbf{G}}_{\mathbf{M}}(R^{\mathbf{G}}_{\mathbf{L}}(\rho_{\Lambda})) \right\rangle \\ &= \left\langle \rho_{\Lambda_{\mathrm{Res}}}, R^{\mathbf{M}}_{\mathbf{L}'}({}^*R^{\mathbf{L}}_{\mathbf{L}'}(\rho_{\Lambda})) \right\rangle \quad = \pm 1 \end{aligned}$$

thanks to the **Mackey formula for** (M, L) and the induction hypothesis (L' is a Levi isomorphic to $\mathbb{F}_q^{\times} \times T \times O^{-\eta}_{2n-2}(\mathbb{F}_q)$).

Explicitly on the symbols:

$$\begin{pmatrix} \cdots & \circ & \circ & \bullet & \circ & \bullet & \bullet & \circ & \circ & \bullet & \bullet & \bullet & \cdots \\ \cdots & \circ & \circ & \circ & \circ & \circ & \bullet & \circ & \bullet & \circ & \bullet & \bullet & \cdots \end{pmatrix}$$

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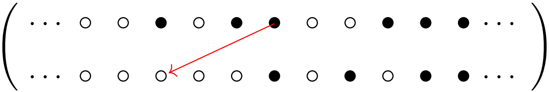

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$$\begin{pmatrix} \cdots & \circ & \circ & \bullet \color{green}{\longrightarrow} & \circ & \bullet & \circ & \circ & \circ & \bullet & \bullet & \bullet & \cdots \\ \cdots & \circ & \circ & \color{red}{\bullet} & \circ & \circ & \bullet & \circ & \bullet & \circ & \bullet & \bullet & \cdots \end{pmatrix}$$

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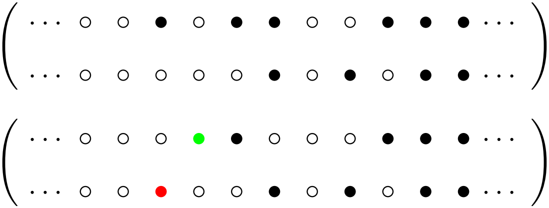

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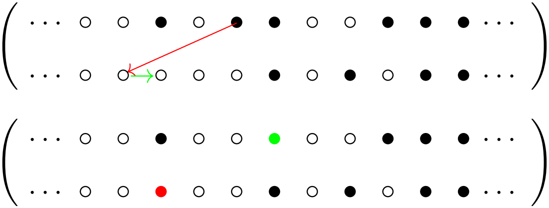
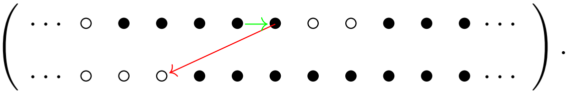


For which cases we cannot find such a Λ_{Res}? One of the cases occurs when Λ and Λ_{Ind} are of the form



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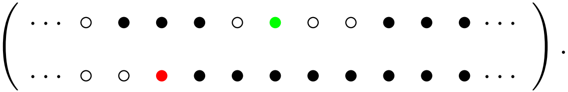

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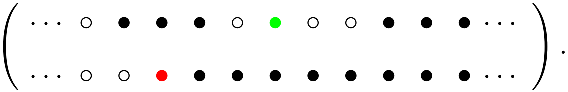
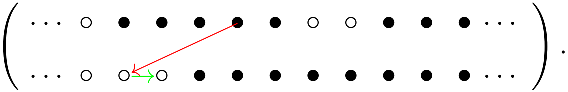


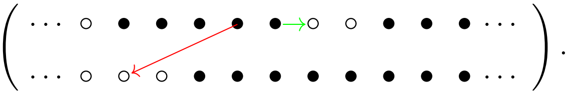
We can obtain Λ_{Res} doing



And both symbols cancel out when we apply the restriction to M, due to the signs on the formula in SO_{2n}.

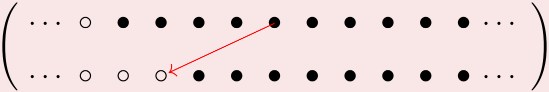
Good news: Both induced symbols appear the same way in $R_L^G(\rho_\Lambda)$.

Solution: For the second symbol, there is a good Λ_{Res}:



So this will determine Λ_{Ind}.

But with the symbol



this doesn't work.

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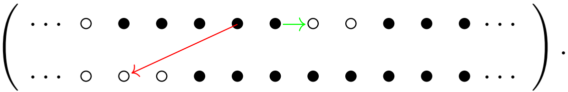

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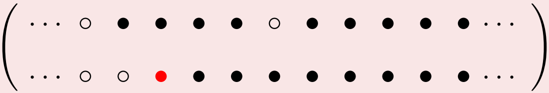

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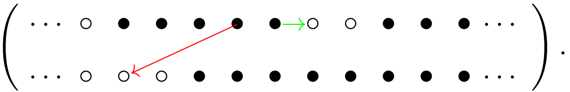

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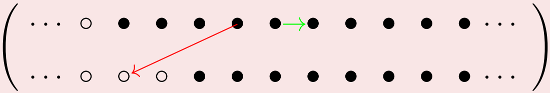

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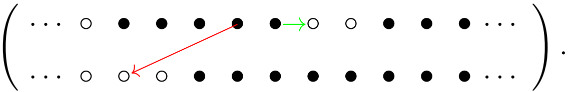

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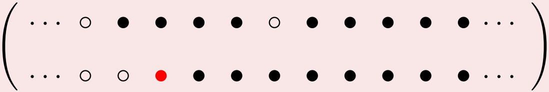

this doesn't work.

Idea: Do the same process with $M' \cong T' \times O^{-\eta}_{2n+2e-2}(\mathbb{F}_q)$ with $|T'| = q+1$, we add 1-cohooks instead of hooks. **But we need other Mackey formula.**