A Mackey Formula for non connected reductive groups

Sergio David Cia Luvecce

Sorbonne Université

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Mackey Formula :

Let $\mathcal{S}_{\mathbf{G}^\circ}(\mathbf{L}^\circ, \mathbf{M}^\circ) = \{x \in \mathbf{G}^\circ | \mathbf{L}^\circ \cap {}^x\mathbf{M}^\circ \text{ contains a maximal torus}\}$. Then

$$ {}^*R_{\mathbf{L}^\circ}^{\mathbf{G}^\circ} \circ R_{\mathbf{M}^\circ}^{\mathbf{G}^\circ} = \sum_{x \in \mathbf{L}^{\circ F} \backslash \mathcal{S}_{\mathbf{G}^\circ}(\mathbf{L}^\circ, \mathbf{M}^\circ)^F / \mathbf{M}^{\circ F}} R_{\mathbf{L}^\circ \cap {}^x\mathbf{M}^\circ}^{\mathbf{L}^\circ} \circ {}^*R_{\mathbf{L}^\circ \cap {}^x\mathbf{M}^\circ}^{{}^x\mathbf{M}^\circ} \circ \mathrm{ad}(x). $$

This formula is true if $q > 2$, or if $\mathbf{G}^\circ$ has no component of type ${}^2\mathrm{E}_6, \mathrm{E}_7$ or E_8 (Bonnafé-Michel, '10).

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Particular case : $\mathbf{P}^\circ = \mathbf{L}^\circ \ltimes \mathbf{U}$, $\mathbf{Q}^\circ = \mathbf{M}^\circ \ltimes \mathbf{V}$, $\mathbf{P}^\circ$, $\mathbf{Q}^\circ$ F-stable parabolic subgroups.

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$$\mathbb{C}\left[\mathbf{U}^F \backslash \mathbf{G}^{\circ F} / \mathbf{V}^F\right] \cong \bigoplus_{x \in \mathbf{L}^{\circ F} \backslash \mathcal{S}_{\mathbf{G}^\circ}(\mathbf{L}^\circ, \mathbf{M}^\circ)^F / \mathbf{M}^{\circ F}} \mathbb{C}\left[\mathbf{L}^{\circ F} x \mathbf{M}^{\circ F}\right],$$

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And $\mathbb{C}\left[\mathbf{L}^{\circ F} x \mathbf{M}^{\circ F}\right]$ isomorphic to

$$\mathbb{C}\left[\mathbf{L}^{\circ F} / (\mathbf{L}^\circ \cap {}^x\mathbf{V})^F \times_{\mathbf{L}^{\circ F} \cap {}^x\mathbf{M}^{\circ F}} ({}^x\mathbf{M}^\circ \cap \mathbf{U})^F \backslash {}^x\mathbf{M}^{\circ F}\right].$$

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Non-connected reductive groups.

$\mathbf{G}$ reductive group, $\mathbf{G}^\circ$ neutral component, F Frobenius endomorphism. Suppose $\mathbf{G}/\mathbf{G}^\circ = \langle \sigma \rangle$ with σ semisimple quasi-central and F-stable.

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We take $\mathbf{P}^\circ = \mathbf{L}^\circ \ltimes \mathbf{U}$, $\mathbf{Q}^\circ = \mathbf{M}^\circ \ltimes \mathbf{V}$. We define

- $\mathbf{P} := N_{\mathbf{G}}(\mathbf{P}^\circ)$, $\mathbf{Q} := N_{\mathbf{G}}(\mathbf{Q}^\circ)$ "Parabolic" subgroups of $\mathbf{G}$
- $\mathbf{L} := N_{\mathbf{G}}(\mathbf{L}^\circ, \mathbf{P}^\circ)$, $\mathbf{M} := N_{\mathbf{G}}(\mathbf{M}^\circ, \mathbf{Q}^\circ)$ "Levi" subgroups.

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Deligne-Lusztig functors for $\mathbf{L}$ and $\mathbf{G}$: same as in $\mathbf{G}^\circ$. Variety :

$$\{g \in \mathbf{G} | g^{-1}F(g) \in \mathbf{U}\}$$

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$$\{g \in \mathbf{G} | g^{-1}F(g) \in \mathbf{U}\}$$

If $\mathbf{L}$, $\mathbf{P}$ are F-stable, then the variety is $\mathbf{G}^F/\mathbf{U}^F$.

Formula for components of type $\mathbf{G}^\circ\sigma$.

We suppose that $\mathbf{L}^\circ, \mathbf{M}^\circ, \mathbf{P}^\circ, \mathbf{Q}^\circ$ are F-stable and that $\sigma \in \mathbf{L}, \mathbf{M}$. Therefore $\mathbf{L} = \mathbf{L}^\circ \rtimes \langle\sigma\rangle$, $\mathbf{L}^F = \mathbf{L}^{\circ F} \rtimes \langle\sigma\rangle$, etc.

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Deligne-Lusztig functors restrict to class functions supported on $\mathbf{G}^{\circ F}\sigma$. We call them $R_{\mathbf{M}^\circ\sigma}^{\mathbf{G}^\circ\sigma}$, ${}^*R_{\mathbf{L}^\circ\sigma}^{\mathbf{G}^\circ\sigma}$.

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Theorem (Digne-Michel, '94)

We have that

$$ {}^*R^{\mathbf{G}^\circ\sigma}_{\mathbf{L}^\circ\sigma} \circ R^{\mathbf{G}^\circ\sigma}_{\mathbf{M}^\circ\sigma} = \sum_{\substack{x \in \mathbf{L}^{\circ F} \backslash \mathcal{S}_{\mathbf{G}^\circ}(\mathbf{L}^\circ, \mathbf{M}^\circ)^F / \mathbf{M}^{\circ F} \\ x \ \ \sigma\text{-stable}}} R^{\mathbf{G}^\circ\sigma}_{(\mathbf{L}^\circ \cap {}^x\mathbf{M}^\circ)\sigma} \circ {}^*R^{{}^x\mathbf{M}^\circ\sigma}_{(\mathbf{L}^\circ \cap {}^x\mathbf{M}^\circ)\sigma} \circ \operatorname{ad} x. $$

Global formula for split "Levi" subgroups and $\mathbf{G}/\mathbf{G}^\circ$ cyclic.

Same setting as before.

Let $\mathcal{S}_{\mathbf{G}}(\mathbf{L},\mathbf{M}) := \{x \in \mathbf{G} | \mathbf{L} \cap {}^x\mathbf{M}$ contains a maximal torus of $\mathbf{G}^\circ\}$.

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Then, as $(\mathbf{L}^{\circ F}, \mathbf{M}^{\circ F})$-bimodules :

$$\mathbb{C}\left[\mathbf{U}^F \backslash \mathbf{G}^F / \mathbf{V}^F\right] \cong \bigoplus_{x \in \mathbf{L}^{\circ F} \backslash \mathcal{S}_{\mathbf{G}}(\mathbf{L},\mathbf{M})^F / \mathbf{M}^{\circ F}} \mathbb{C}\left[\mathbf{L}^{\circ F} x \mathbf{M}^{\circ F}\right].$$

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We have $x \in \mathcal{S}_{\mathbf{G}}(\mathbf{L},\mathbf{M}) \Rightarrow \sigma x, x\sigma \in \mathcal{S}_{\mathbf{G}}(\mathbf{L},\mathbf{M})$.

As an $(\mathbf{L}^F, \mathbf{M}^F)$-bimodule, the terms corresponding to $\sigma^i x \sigma^j$ form a single factor $\mathbf{L}^F x \mathbf{M}^F$.

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- $\exists x$ such that $\sigma x = x\sigma$, the terms for $x, x\sigma$ form $\mathbf{L}^F x \mathbf{M}^F$.
- $\mathbf{L} \cap {}^x\mathbf{M} = \mathbf{L}^\circ \cap {}^x\mathbf{M}^\circ \rtimes \langle\sigma\rangle$ is a "Levi" of $\mathbf{G}$.

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$$(l, m), (l\sigma, m), (l, m\sigma), (l\sigma, m\sigma)$$

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Therefore, as $(\mathbf{L}^F, \mathbf{M}^F)$-sets,

$$\mathbf{L}^F x \mathbf{M}^F \cong \mathbf{L}^F/(\mathbf{L} \cap {}^x\mathbf{V})^F \times_{\mathbf{L}^F \cap {}^x\mathbf{M}^F} ({}^x\mathbf{M} \cap \mathbf{U})^F \backslash {}^x\mathbf{M}^F.$$

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- We already have :

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We proved

$$\mathbb{C}\left[\mathbf{L}^F x \mathbf{M}^F\right] \cong \mathbb{C}\left[\mathbf{L}^F/(\mathbf{L}\cap {}^x\mathbf{V})^F \times_{\mathbf{L}^F\cap {}^x\mathbf{M}^F} ({}^x\mathbf{M}\cap \mathbf{U})^F\backslash {}^x\mathbf{M}^F\right]$$

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On the first case, it is the functor $R^{\mathbf{L}}_{\mathbf{L}\cap {}^x\mathbf{M}} \circ {}^*R^{{}^x\mathbf{M}}_{\mathbf{L}\cap {}^x\mathbf{M}}$.

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Theorem (C.)

For $\mathbf{G}/\mathbf{G}^\circ = \langle\sigma\rangle$, $\mathbf{L}$, $\mathbf{M}$ split and containing σ :

$$ {}^*R^{\mathbf{G}}_{\mathbf{L}} \circ R^{\mathbf{G}}_{\mathbf{M}} = \sum_{x \in \mathbf{L}^F \backslash \mathcal{S}_{\mathbf{G}}(\mathbf{L},\mathbf{M})^F / \mathbf{M}^F} R^{\mathbf{L}}_{\mathbf{L}\cap {}^x\mathbf{M}} \circ {}^*R^{{}^x\mathbf{M}}_{\mathbf{L}\cap {}^x\mathbf{M}} \circ \mathrm{ad}(x). $$

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$|\sigma|$ arbitrary : We have $\mathbf{L} \cap {}^x\mathbf{M} = \mathbf{L}^\circ \cap {}^x\mathbf{M}^\circ \rtimes \langle \sigma^i \rangle$ for an $i \in \{0, \ldots, |\sigma| - 1\}$.

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$|\sigma|$ arbitrary : We have $\mathbf{L} \cap {}^x\mathbf{M} = \mathbf{L}^\circ \cap {}^x\mathbf{M}^\circ \rtimes \langle\sigma^i\rangle$ for an $i \in \{0, \ldots, |\sigma| - 1\}$.

$$ \mathbf{L}^F x \mathbf{M}^F \cong \mathbf{L}^F/(\mathbf{L} \cap {}^x\mathbf{V})^F \times_{\mathbf{L}^F \cap {}^x\mathbf{M}^F} ({}^x\mathbf{M} \cap \mathbf{U})^F \backslash {}^x\mathbf{M}^F $$

.

$\mathbf{L} \cap {}^x\mathbf{M}$ "Levi" of $\mathbf{G}^\circ \rtimes \langle\sigma^i\rangle$, we induce to $\mathbf{L}^\circ \rtimes \langle\sigma^i\rangle$, and restrict from ${}^x\mathbf{M}^\circ \rtimes \langle\sigma^i\rangle$.

Mackey Formula, general case.

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We define $\mathbf{L}_x = \mathbf{L} \cap \mathbf{G}_x$, $\mathbf{M}_x = \mathbf{M} \cap \mathbf{G}_x$ and $R^{\mathbf{L}}_{\mathbf{L}\cap {}^x\mathbf{M}} = \mathrm{Ind}^{\mathbf{L}^F}_{\mathbf{L}_x^F} \circ R^{\mathbf{L}_x}_{\mathbf{L}\cap {}^x\mathbf{M}}$, ${}^*R^{{}^x\mathbf{M}_x}_{\mathbf{L}\cap {}^x\mathbf{M}} = {}^*R^{{}^x\mathbf{M}_x}_{\mathbf{L}\cap {}^x\mathbf{M}} \circ \mathrm{Res}^{{}^x\mathbf{M}^F}_{{}^x\mathbf{M}_x^F}$.

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Theorem (C.)

$${}^*R^{\mathbf{G}}_{\mathbf{L}} \circ R^{\mathbf{G}}_{\mathbf{M}} = \sum_{x \in \mathbf{L}^F \backslash \mathcal{S}_{\mathbf{G}}(\mathbf{L},\mathbf{M})^F / \mathbf{M}^F} R^{\mathbf{L}}_{\mathbf{L}\cap {}^x\mathbf{M}} \circ {}^*R^{{}^x\mathbf{M}}_{\mathbf{L}\cap {}^x\mathbf{M}} \circ \mathrm{ad}(x).$$